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Zone. °	Nos. of Plates.	Difference of *a*	Means.	Difference of *e*.	Means.
65	1274–534	+ ·00023	+ ·00026	+ ·00013	– ·00007
	534–426	+ ·00036		– ·00014	
	426–2270	+ ·00032		– ·00033	
	2280–535	+ ·00013		+ ·00006	
66	443–444	+ ·00026	+ ·00019	+ ·00019	– ·00005
		+ ·00034		+ ·00028	
	444–1236	+ ·00015		– ·00024	
	1236–2290	+ ·00007		– ·00018	
	2290–2238	+ ·00037		– ·00028	
	2279–2308	+ ·00007		– ·00006	
67	2288–2289	+ ·00012	+ 00019	+ ·00002	+ ·00001
	2289–2251	+ ·00024		+ ·00005	
	2251–1416	+ ·00020		– ·00005	
68	2136–2227	+ ·00039	+ ·00029	+ ·00058	+ ·00010
	2227–1397	+ ·00023		+ ·00021	
		+ ·00013		+ ·00001	
	1397–2306	+ ·00006		– ·00016	
		+ ·00024		+ ·00002	
	2057–2058	+ ·00045		– ·00006	
69	1323–2291	+ ·00004	+ ·00004	– ·00002	– ·00002

The plates whose numbers are less than 2227 were taken before the alteration in the adjustment for tilt on 1894 September 17. Examination of the figures in the above table shows that this did not sensibly alter the value of the discordance in question.

A Graphical Method of Solving Kepler's Equation.

By H. C. Plummer.

(*Communicated by H. H. Turner.*)

The prominence which Dr. T. J. J. See in a recent paper published in the *Monthly Notices* gave to the Waterston-Dubois method of solving Kepler's Equation makes it appear likely that another graphical method may have some interest. The question of authorship to which that paper gave rise causes hesitation in claiming originality; but the method which is here described has not, so far as I am aware, been published previously. It is proposed to show a way of finding an approximate solution of the

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equation $E - M = e \sin E$ corresponding to a given value of M, e being $\nless$ ·1 and $\ngtr$ ·9, and also to show how the error in the value of E so found may be discovered very simply by graphical means.

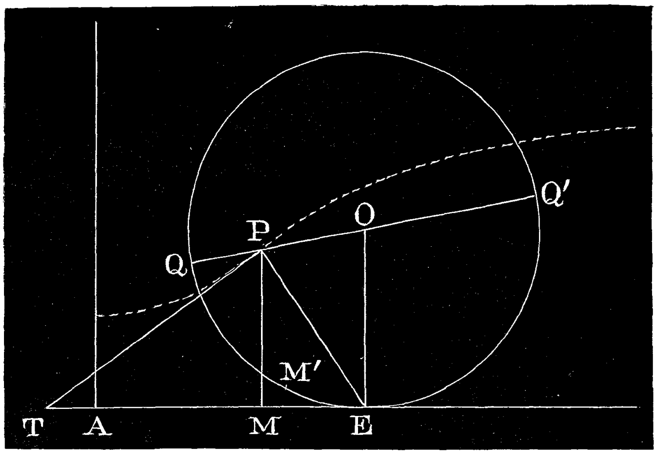

Fig. 1.

Let a circle (centre O) of unit radius *roll* along a straight line, and let a radius OQ be divided at P in the ratio $e : 1 - e$. Let A be the point where Q was originally on the line, M the foot of the ordinate through P, and E the point of contact of the circle at the instant.

Then

$$AM = AE - ME = \text{arc } QE - OP \sin POE.$$

Hence if AM=M, since the radius is of unit length, and $OP = e$, arc QE=E. But if M′ is the point on the circle such that QM′=AM=M, M′ was formerly coincident with the present position of M.

It thus appears that we may take an ordinary semicircular protractor, marking the point P which divides the radius through the zero reading in the ratio $e : 1 - e$. The division will be facilitated if the radius has a convenient scale marked along it. The protractor is placed in contact with a straight ledge at a fixed point, so that the reading at this point is M. The instrument must then be rolled along the ledge until the point P comes on the perpendicular to the ledge at the fixed point. The reading at the new point of contact is the required angle E.

The method just described has the great advantage of possessing the utmost simplicity of construction, requiring no specially prepared curve, and being practically instantaneous in application. On the other hand, it is only possible to obtain very roughly approximate results on account of the difficulty of avoiding an error, amounting to a considerable fraction of a degree, in reading off the point of the circumference in contact with the ledge. It is therefore necessary to slightly modify the method

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in order to get more refined approximations. For this purpose it must be noticed (1) that since (fig. 1) the arc QE=AE, the readings may be taken along, or parallel to, the base, instead of round the circumference; and (2) that since OE is an ordinate, the abscissa of O may be read instead of the abscissa of E.

Let a board be covered with millimetre paper, one set of lines being parallel to a straight ledge at the side of the board. Let the length of the base (corresponding to 180°) be 450 mm. (say), so that 2·5 mm. correspond to a degree. Now make a semicircle of some material, stout cardboard, for instance, whose curved boundary is also 450 mm., so that its radius is

$$\frac{450}{\pi}\text{ mm.} = 143{\cdot}24\text{ mm.}$$

If the disc is covered with millimetre paper, with one set of lines perpendicular to one of the bounding radii, it is easy to find the point P which divides the radius in the ratio $e : 1-e$ very accurately. If the semicircle is now placed with this radius perpendicular to the base at one extremity, and then rolled along the base until P has the abscissa M (the extremity of the base being the origin of coordinates), the abscissa of the centre is E, and can be read off with no greater error than 0°·1, provided *no slipping* of the semicircle has been allowed to take place.

Let E_0 be the approximation to E thus found, and let $M_0 = E_0 - e \sin E_0$. Hence $\Delta M = M - M_0$ can be found at once. Now

$$\begin{aligned} E &= f(M) = f(M_0 + \Delta M) \\ &= f(M_0) + \Delta M f'(M_0) + \frac{(\Delta M)^2}{2} f''(M_0) + \ldots \\ &= E_0 + \frac{\Delta M}{1 - e\cos E_0} - \frac{(\Delta M)^2}{2} \cdot \frac{e \sin E_0}{(1 - e\cos E_0)^2} \cdot \frac{1}{1 - e\cos E_0} + \ldots \end{aligned}$$

Hence neglecting small quantities of order higher than the first,

$$E - E_0 = \Delta E = \frac{\Delta M}{1 - e\cos E_0}.$$

Now $1 - e \cos E_0$ can be read off at once, for referring to fig. 1,

$$1 - e\cos E_0 = OE - OP\cos POE = PM$$

and it is therefore only necessary to read off the ordinate of P. The facility with which this correcting factor can be found seems to show that this method may be of considerable value in those cases where a second approximation by graphical means is desired. In practice it will probably be found best to treat the new approximation to E in the same way if approximations still closer are desired. It is, however, perhaps worthy of note that the calculation of the term in the above series involving a small

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quantity of the second order is made very simple by the graphical method. For suppose the perpendicular at P to PE meets the base in T (fig. 1); then

$$\mathrm{TM} = \mathrm{PM}^2/\mathrm{ME} = (1 - e\cos \mathrm{E})^2/e \sin \mathrm{E}$$

and the coefficient need only be estimated roughly, since it multiplies a very small quantity.

The roulette traced by P is of course a prolate trochoid, to which PE is the normal at P, and TM is a sub-tangent. The instant when P falls on a given ordinate is always well marked, since the maximum inclination to the base of the tangent to the trochoid is $\sin^{-1}e$. This fact, which can be seen very easily, makes it clear that the error in reading off E is sensibly the same for all values of M, as well as for all values of e, for $\sin^{-1}e$ never exceeds 65°. This is an important point in favour of what may be called the trochoid method, for under certain circumstances the use of the curve of sines cannot give a very good result on account of the approach to parallelism of the curve and the line $x - ey = \mathrm{M}$ at the point of intersection.

The method which has been here described is fully as general in its application as that of Waterston, and though I have had no opportunity of making an extended practical use of either method for the purposes of comparison, there seems to be no reason to anticipate any other than a successful trial for the trochoid method. The *slipping* of the disc along the base is the one thing to be feared, but this can be avoided by careful use and by giving a rough edge to the disc. Besides being equally well adapted to cases in which only rough approximations are needed, or in which the highest degree of accuracy attainable by graphical means is required, the trochoid method promises to be greatly superior when a limited use only is expected, in which case the trouble of constructing a reasonably accurate curve of sines would render the older method quite impracticable.

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